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Turbulence measurements with inclined hot-wires

Part 2. Hot-wire response equations

By F. H. CHAMPAGNE†
Department of Chemical Engineering, University of Washington
and Boeing Scientific Research Laboratories

AND C. A. SLEICHER
Department of Chemical Engineering, University of Washington

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Hot-wire response equations to include the effects of the tangential velocity component as well as the non-linearities caused by high intensity turbulence are derived for linearized constant temperature operation. For low intensity turbulence similar equations are derived for constant current operation. The equations are applied to an X-wire array to determine the errors in selected turbulence quantities which arise from the assumption of cosine law cooling. The error depends upon the quantity measured, the method of operation, and ℓ/d. For $\ell/d = 200$ the error ranges from 0 to 17 %.

The results of the heat transfer and temperature distribution measurements presented in part 1 indicate that an inclined hot-wire is sensitive to the tangential velocity component along the wire. This sensitivity must be taken into consideration when interpreting data from inclined hot-wires. A wire is best calibrated directly at all desired yaw angles, but it may be calibrated only normal to the flow and then corrected for yaw with the equations derived below.

Define intrinsic co-ordinates s, n, t with $\mathbf{e}_s$ as a unit vector tangent to the mean streamline, while $\mathbf{e}_n$ and $\mathbf{e}_t$ coincide with the principal normal and binormal directions, respectively, of the mean streamline as shown in figure 1. Let Q_s be the resulting mean velocity and q_s, q_n and q_t be the velocity component fluctuations in the s, n, t directions, respectively. The notation Q and q rather than the conventional $\overline{U}$ and u for the velocity components is used here to coincide with the notation used recently by Rose (1962).

The magnitude of the instantaneous velocity vector is (figure 1)

$$Q_I = [(Q_s+q_s)^2+q_n^2+q_t^2]^{\frac{1}{2}}. \tag{1}$$

The instantaneous effective cooling velocity is given by

$$Q_E^2 = Q_I^2[\cos^2\beta_3+k^2\sin^2\beta_3], \tag{2}$$

where β_3 is the angle between the instantaneous velocity vector and the normal to the wire axis. For this derivation the wire is located in the (s, n)-plane although the results can be readily modified for the case of the wire located in the (s, t)-plane.

† Present address: Boeing Scientific Research Laboratories, Seattle, Washington.

β_3 can be expressed in terms of α, the angle between the normal to the wire and the mean flow direction, and the velocity components as follows. Applying the cosine law of trignometry (see figure 1) yields:

$$-\sin\beta_3 = \left[\cos^2\alpha + \frac{2\sin\alpha\cos\alpha\tan\beta_2}{\cos\beta_4} + \frac{\sin^2\alpha\tan^2\beta_2}{\cos^2\beta_4} + \sin^2\alpha\tan^2\beta_4 - 1 - \frac{\sin^2\alpha}{\cos^2\beta_2\cos^2\beta_4}\right]\frac{\cos\beta_2\cos\beta_4}{2\sin\alpha}. \quad (3)$$

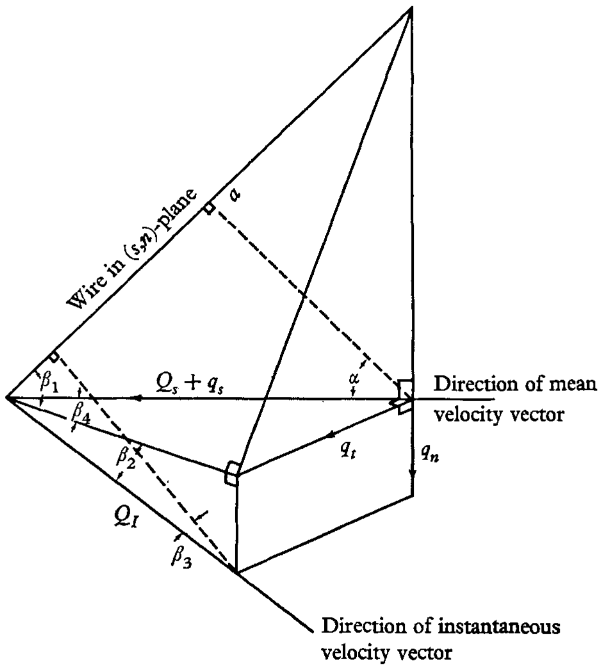

FIGURE 1. Velocity component diagram.

The angles β_2 and β_4 are defined by

$$\left.\begin{aligned} \sin\beta_2 &= q_n[(Q_s+q_s)^2+q_t^2+q_n^2]^{-\frac{1}{2}}, \quad & \cos\beta_2 &= [(Q_s+q_s)^2+q_t^2][(Q_s+q_s)^2+q_t^2+q_n^2]^{-\frac{1}{2}},\\ \sin\beta_4 &= q_t[(Q_s+q_s)^2+q_t^2]^{-\frac{1}{2}}, & \cos\beta_4 &= [Q_s+q_s][(Q_s+q_s)^2+q_t^2]^{-\frac{1}{2}}. \end{aligned}\right\} \quad (4)$$

Introducing (4) into (3) and rearranging gives

$$-\sin\beta_3 = [-\sin\alpha+\cos\alpha q_n(Q_s+q_s)^{-1}][Q_s+q_s][(Q_s+q_s)^2+q_t^2+q_n^2]^{-\frac{1}{2}}. \quad (5)$$

Squaring (5) gives

$$\sin^2\beta_3 = \left[\left(1+\frac{q_s}{Q_s}\right)^2+\frac{q_n^2+q_t^2}{Q_s^2}\right]^{-1}\left[\sin^2\alpha\left(1+\frac{q_s}{Q_s}\right)^2 - 2\sin\alpha\cos\alpha\left(\frac{q_n}{Q_s}+\frac{q_nq_s}{Q_s^2}\right)+\cos^2\alpha\frac{q_n^2}{Q_s^2}\right]. \quad (6)$$

The denominator of this equation may be expanded in a power series to give

$$\left[\left(1+\frac{q_s}{Q_s}\right)^2+\frac{q_n^2+q_t^2}{Q_s^2}\right]^{-1} = 1-2\frac{q_s}{Q_s}+3\frac{q_s^2}{Q_s^2}-\frac{q_t^2+q_n^2}{Q_s^2}-4\frac{q_s^3}{Q_s^3} + 4\frac{q_s q_t^2+q_s q_n^2}{Q_s^3}+\text{4th order terms.} \qquad (7)$$

Substituting (7) into (6) and collecting terms gives

$$\sin^2\beta_3 = \left[-\frac{q_t^2+q_n^2}{Q_s^2}+2\frac{q_s q_t^2+q_s q_n^2}{Q_s^2}\right]\sin^2\alpha+\sin^2\alpha+\left[\frac{q_n^2}{Q_s^2}-2\frac{q_s q_n^2}{Q_s^3}\right]\cos^2\alpha - 2\sin\alpha\,\cos\alpha\left[\frac{q_n}{Q_s}-\frac{q_n q_s}{Q_s^2}-\frac{q_n q_t^2}{Q_s^3}+\frac{q_n q_s^2}{Q_s^3}-\frac{q_n^3}{Q_s^3}\right], \qquad (8)$$

which agrees with the result of a similar derivation presented by Heskestad (1963). Thus,

$$\cos^2\beta_3+k^2\sin^2\beta_3 = \cos^2\alpha\left[1+k^2\tan^2\alpha+(k^2-1)\left\{\frac{1}{\cos^2\alpha}\left(-\frac{q_t^2+q_n^2}{Q_s^2}+2\frac{q_s q_t^2+q_s q_n^2}{Q_s^3}\right) + \frac{q_t^2+2q_n^2}{Q_s^2}-2\frac{q_s q_t^2+2q_s q_n^2}{Q_s^3}-2\tan\alpha\left(\frac{q_n}{Q_s}-\frac{q_n q_s}{Q_s^2}-\frac{q_n q_t^2}{Q_s^2}+\frac{q_n q_s^2}{Q_s^3}-\frac{q_n^3}{Q_s^3}\right)\right\}\right]. \qquad (9)$$

Now

$$Q_I^2 = Q_s^2\left(1+2\frac{q_s}{Q_s}+\frac{q_s^2+q_n^2+q_t^2}{Q_s^2}\right).$$

Therefore,

$$Q_E^2(\alpha) = Q_s^2\cos^2\alpha\left[1+2\frac{q_s}{Q_s}+\frac{q_s^2-q_n^2}{Q_s^2}+\frac{1}{\cos^2\alpha}\left[\frac{q_t^2+q_n^2}{Q_s^2}\right]+2\tan\alpha\left[\frac{q_n}{Q_s}+\frac{q_n q_s}{Q_s^2}\right] + k^2\left\{\frac{q_t^2+2q_n^2}{Q_s^2}-\frac{1}{\cos^2\alpha}\left[\frac{q_t^2+q_n^2}{Q_s^2}\right]-2\tan\alpha\left[\frac{q_n}{Q_s}+\frac{q_n q_s}{Q_s^2}\right] + \tan^2\alpha\left[1+2\frac{q_s}{Q_s}+\frac{q_s^2+q_n^2+q_t^2}{Q_s^2}\right]\right\}\right]. \qquad (10)$$

Next, a power series expansion to obtain Q_E gives

$$\begin{aligned}
Q_E(\alpha) = Q_s\cos\alpha\Big[&1+k^2\tfrac{1}{2}\tan^2\alpha-k^4\tfrac{1}{8}\tan^4\alpha+(1+k^2\tfrac{1}{2}\tan^2\alpha-k^4\tfrac{1}{8}\tan^4\alpha)\,q_s/Q_s\\
&+(\tan\alpha-k^2\tan\alpha(1+\tfrac{1}{2}\tan^2\alpha)+k^4\tfrac{1}{2}\tan^3\alpha(1+\tfrac{3}{4}\tan^2\alpha))\,q_n/Q_s\\
&+\left(\frac{1}{2\cos^2\alpha}-k^2\frac{\tan^2\alpha}{4\cos^2\alpha}+\tfrac{3}{16}k^4\frac{\tan^4\alpha}{\cos^2\alpha}\right)q_t^2/Q_s^2-\tfrac{3}{8}k^4\tan^4\alpha q_s^2/Q_s^2\\
&+(k^2(\tfrac{1}{2}+\tan^2\alpha(1+\tfrac{1}{2}\tan^2\alpha))-\tfrac{3}{2}k^4\tan^2\alpha(\tfrac{1}{2}+\tan^2\alpha+\tfrac{5}{8}\tan^4\alpha))\,q_n^2/Q_s^2\\
&-k^4\tan^4\alpha q_s^3/Q_s^3+(-k^2(\tfrac{1}{2}+\tan^2\alpha+\tfrac{1}{2}\tan^4\alpha)\\
&+k^4(\tfrac{3}{4}\tan^2\alpha+\tfrac{3}{4}\tan^4\alpha+\tfrac{9}{2}\tan^6\alpha))\,q_s q_n^2/Q_s^3\\
&+k^4(6\tan^3\alpha-\tfrac{9}{4}\tan^5\alpha)\,q_n q_s^2/Q_s^3\\
&+\left(k^2\frac{\tan^2\alpha}{4\cos^2\alpha}+k^4\tan^4\alpha\left(\tfrac{3}{4}\tan^2\alpha+\frac{3}{\cos^2\alpha}\right)\right)q_s q_t^2/Q_s^3\\
&+\left(k^2\frac{\tan\alpha}{2\cos^2\alpha}(1+\tfrac{3}{2}\tan^2\alpha)+k^4\left(\tfrac{3}{4}\tan^3\alpha-\frac{3\tan^3\alpha}{2\cos^2\alpha}+3\frac{\tan^5\alpha}{\cos^2\alpha}\right)\right)q_n q_t^2/Q_s^3\\
&+(-k^2\tan\alpha(\tfrac{1}{2}+\tan^2\alpha+\tfrac{1}{2}\tan^4\alpha)+k^4(\tfrac{1}{2}\tan\alpha+\tfrac{15}{4}\tan^3\alpha+\tfrac{15}{4}\tan^5\alpha\\
&-\tfrac{1}{16}\tan^7\alpha))\,q_n^3/Q_s^3\Big] \qquad (11)
\end{aligned}$$

to third order in the turbulence terms and to fourth order in k. This expression is sufficient for most applications as the third-order turbulence terms are often negligible and k rarely exceeds 0·2.

If consideration is restricted to low intensity turbulence for clarity of presentation, the second- and third-order terms in (11) can be neglected. This gives

$$Q_E(\alpha) = Q_s \cos\alpha[1+\tfrac{1}{2}k^2\tan^2\alpha - \tfrac{1}{8}k^4\tan^4\alpha + (1+\tfrac{1}{2}k^2\tan^2\alpha - \tfrac{1}{8}k^4\tan^4\alpha)\,q_s/Q_s + (\tan\alpha - k^2\tan\alpha(1+\tfrac{1}{2}\tan^2\alpha) + \tfrac{1}{2}k^4\tan^3\alpha(1+\tfrac{3}{4}\tan^2\alpha))\,q_n/Q_s]. \quad (12)$$

Define $q_E(\alpha)$ as the effective fluctuating velocity component, i.e. the fluctuating part of $Q_E(\alpha)$. Then (12) yields

$$q_E(\alpha) = Q_s \cos\alpha[(1+\tfrac{1}{2}k^2\tan^2\alpha - \tfrac{1}{8}k^4\tan^4\alpha)\,q_s/Q_s + (\tan\alpha - k^2\tan\alpha(1+\tfrac{1}{2}\tan^2\alpha) + \tfrac{1}{2}k^4\tan^3\alpha(1+\tfrac{3}{4}\tan^2\alpha))\,q_n/Q_s]. \quad (13)$$

Linearized constant temperature operation

The output voltage of a linearized constant temperature hot-wire anemometer is given by

$$E(t) = \bar{E} + e(t) = KQ_E(t), \quad (14)$$

where Q_E is given by (12). Applying (12), (13) and (14) to an ideal X-wire array (shown in figure 2) gives

$$\bar{E}_1 = \bar{E}_2 = 2^{-\frac{1}{2}}KQ_s[1+\tfrac{1}{2}k^2 - \tfrac{1}{8}k^4], \quad (15)$$

$$\frac{\overline{q_s^2}}{Q_s^2} = \frac{1}{4}\overline{\left(\frac{e_1(t)}{\bar{E}_1} + \frac{e_2(t)}{\bar{E}_2}\right)^2}, \quad (16)$$

$$\frac{\overline{q_s q_n}}{Q_s^2} = \left(\frac{1+k^2}{1-k^2}\right)\frac{1}{4}\left(\frac{\overline{e_1^2(t)}}{\bar{E}_1^2} - \frac{\overline{e_2^2(t)}}{\bar{E}_2^2}\right), \quad (17)$$

$$\frac{\overline{q_n^2}}{Q_s^2} = \left(\frac{1+k^2}{1-3k^2+4k^4}\right)\frac{1}{4}\overline{\left(\frac{e_1(t)}{\bar{E}_1} - \frac{e_2(t)}{\bar{E}_2}\right)^2}, \quad (18)$$

or

$$\left(\frac{\overline{q_s^2}}{Q_s^2}\right)_c = \left(\frac{\overline{q_s^2}}{Q_s^2}\right)_N, \quad (19)$$

$$\left(\frac{\overline{q_s q_n}}{Q_s^2}\right)_c = \left(\frac{1+k^2}{1-k^2}\right)\left(\frac{\overline{q_s q_n}}{Q_s^2}\right)_N, \quad (20)$$

$$\left(\frac{\overline{q_n^2}}{Q_s^2}\right)_c = \left(\frac{1+k^2}{1-3k^2+4k^4}\right)\left(\frac{\overline{q_n^2}}{Q_s^2}\right)_N, \quad (21)$$

where the subscript c denotes corrected for the effect of the tangential velocity component and the subscript N denotes normal component cooling only.

Constant current operation

The constant current method operation is useful only in low intensity turbulence because of the non-linear character of the response to fluctuations in magnitude and direction of the fluid velocity. The derivation will therefore be restricted to low intensity turbulence, for which the linearized approximation

$$\frac{I^2\bar{R}_e}{\bar{R}_a - R_a}\left(1 - \frac{R_a}{\bar{R}_e - R_a}\frac{r_e}{\bar{R}_e}\right) = A_a + B_a(Q_E(\alpha))^{\frac{1}{2}} \tag{22}$$

is valid. $\bar{R}_e$ and r_e stand for 'equilibrium resistances' as defined by Corrsin (1963). R_a is the resistance of the wire evaluated at ambient temperature, T_a.

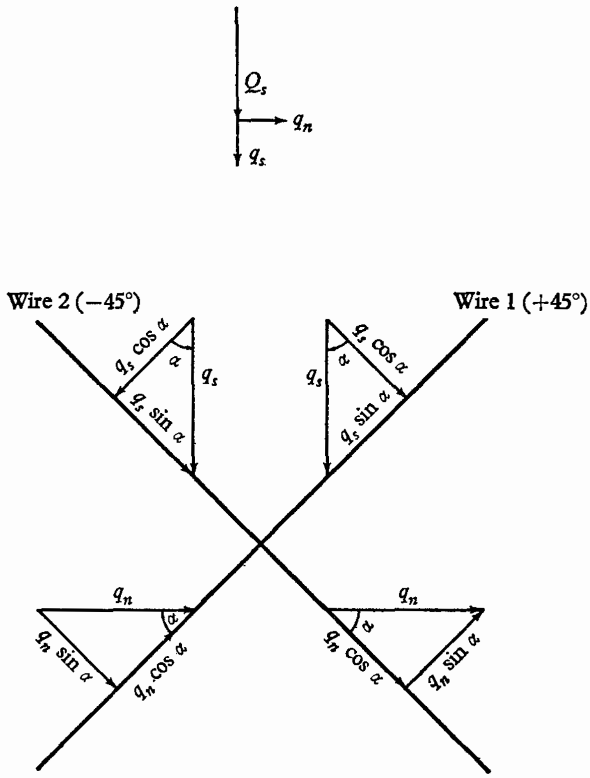

FIGURE 2. Schematic diagram of turbulent velocity components on an X-wire probe.

The mean and fluctuating parts of the voltage are given by

$$\bar{E} = I\bar{R}_e, \quad e(t) = Ir_e(t). \tag{23, 24}$$

The square root of the effective cooling velocity may be obtained from power series expansion of (10). If this expression for $Q_E^{\frac{1}{2}}(\alpha)$ is combined with (22), (23) and (24), the following results are obtained:

$$\bar{E}_1 = \bar{E}_2 = (\bar{R}_e - R_a)I^{-1}[A_a + B_a(Q_s\cos\alpha)^{\frac{1}{2}}(1 + \tfrac{1}{4}k^2 - \tfrac{3}{32}k^4)], \tag{25}$$

$$\left(\frac{\overline{q_s^2}}{Q_s^2}\right)_c = \left(\frac{\overline{q_s^2}}{Q_s^2}\right)_N, \tag{26}$$

$$\left(\frac{\overline{q_s q_n}}{Q_s^2}\right)_c = \left(\frac{1 + \frac{1}{2}k^2 - \frac{1}{8}k^4}{1 - 2k^2 + \frac{89}{16}k^4}\right)\left(\frac{\overline{q_s q_n}}{Q_s^2}\right)_N, \tag{27}$$

$$\left(\frac{\overline{q_n^2}}{Q_s^2}\right)_c = \left(\frac{1 + \frac{1}{2}k^2 - \frac{1}{8}k^4}{1 + \frac{1}{2}k^2 + \frac{61}{16}k^4}\right)\left(\frac{\overline{q_n^2}}{Q_s^2}\right)_N. \tag{28}$$

The error in turbulence measurements caused by assuming normal component or cosine law cooling can now be assessed. With a value of $k = 0{\cdot}20$, which corresponds to an $\ell/d \approx 200$, it follows from (19), (20) and (21) that for linearized constant temperature operation

$$\left(\frac{\overline{q_s^2}}{Q_s^2}\right)_c = \left(\frac{\overline{q_s^2}}{Q_s^2}\right)_N, \quad \left(\frac{\overline{q_s q_n}}{Q_s^2}\right)_c = 1{\cdot}08\left(\frac{\overline{q_s q_n}}{Q_s^2}\right)_N, \quad \left(\frac{\overline{q_n^2}}{Q_s^2}\right)_c = 1{\cdot}17\left(\frac{\overline{q_n^2}}{Q_s^2}\right)_N.$$

Similarly, for constant current operation, there results

$$\left(\frac{\overline{q_s^2}}{Q_s^2}\right)_c = \left(\frac{\overline{q_s^2}}{Q_s^2}\right)_N, \quad \left(\frac{\overline{q_s q_n}}{Q_s^2}\right)_c = 1{\cdot}10\left(\frac{\overline{q_s q_n}}{Q_s^2}\right)_N, \quad \left(\frac{\overline{q_n^2}}{Q_s^2}\right)_c = 1{\cdot}16\left(\frac{\overline{q_n^2}}{Q_s^2}\right)_N.$$

Although errors from many sources arise in turbulence measurements, the above errors are not negligible and should be taken into account when making careful turbulence measurements.

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